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Kantowski-Sachs Cosmological Model in General Theory of Relativity

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Abstract The field equations for perfect fluid coupled with massless scalar field are solved with two conditions $p=\rho$ and $R=AS^n$ for Kantowski-Sachs space time in general theory of relativity. Various physical and geometrical properties of the model have also been discussed.

Keywords Perfect fluid · Massless scalar field · Kantowski-Sachs space-time

1 Introduction

Any physical theory can be studied easily through the exact solutions of its mathematical structure. Therefore, the exact solutions of relativistic model carry important role than those obtained through approximation scheme and numerical computation. Moreover, the relativists use various symmetries to get physical viable information from the complicated structure of the field equations in Einstein's theory. The gravitational effects of cylindrically symmetric interacting massless scalar fields are a subject of current interest because of their possible applications to nuclear physics. The case of coupled source free electromagenetic fields and stiff fluid distributions equivalent to massless scalar fields was investigated by [1]. The origin of structure in the universe is one of the greatest cosmological mysteries even today. The present day observations indicate that the universe at large scale is homogeneous and isotropic and it is accelerating phase of the universe (recently detected experimentally) [22]. It is well known that exact solutions of general theory of relativity for homogeneous space times belongs to either Bianchi types or Kantowski-Sachs [24].

Weber [2, 3] had done a qualitative study of the Kantowski and Sachs [15] cosmological models. Lorcoz [4], Gron [6], Matravers [7], Krori et al. [8], Dabrowski [9], Li and Hao [10] have also studied cosmological models for the Kantowski-Sachs space time. Lorenz [5] has obtained exact Kantowski-Sachs vacuum models in Brans-Dicke [11] theory, while Sing and Agrawal [12] discussed Kantowski-Sachs type models in Saez and Ballester [21]

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scalar tensor theory. Recently, Reddy [13, 14] presented a string cosmological models in Brans-Dicke and Saez-Ballester scalar-tensor theories of gravitation.

Also the massless scalar field in relativistic mechanics yields some significant results regarding both the singularities involved and Mach's principle. Panigrahi and Sahu [20] studied a micro and macro cosmological models in the presence of massless scalar field interacted with perfect fluid.

In this paper, we have studied cosmological model generated by perfect fluid coupled with massless scalar field for Kantowski-Sachs space time in general theory of relativity. For solving the field equations, the relation $p = \rho$ and relation between metric potentials $R = AS^n$ are used. Some physical and geometrical properties of the determinate model are also discussed.

2 Field Equations

We consider the Kantowski-Sachs space time metric in the form

$$ds^2 = dt^2 - R^2 dr^2 - S^2(d^2\theta + \mathrm{Sin}^2\theta d^2\phi), \tag{1}$$

where R and S are the functions of time t only.

The relativistic field equations for coupled perfect fluid and massless scalar fields read as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -(T^P_{ij} + T^v_{ij}). \tag{2}$$

The stress energy tensors T^P_{ij} and T^v_{ij} corresponding to perfect fluid and massless scalar fields are given as

$$T^P_{ij} = (\rho + p)u_i u_j - pg_{ij} \tag{3}$$

and

$$T^v_{ij} = v_{,i}\, v_{,j} - \frac{1}{2}g_{ij}v_{,k}\, v^{,k}, \tag{4}$$

where perfect fluid and scalar field v satisfy the relation

$$g_{ij}u^i u^j = 1 \tag{5}$$

and

$$g^{ij}v_{,ij} = 0, \tag{6}$$

where u^i is the four velocity vector of the fluid. p and ρ are the proper pressure and energy density respectively. Comma and semicolon denotes partial and covariant differentiation respectively. By adoption of co-moving coordinates, the field equations (2) for the line element (1) can be written as

$$2\frac{S_{44}}{S} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} = -p - \frac{1}{2}v_4^2, \tag{7}$$

$$\frac{R_{44}}{R} + \frac{S_{44}}{S} + \frac{R_4 S_4}{RS} = -p - \frac{1}{2}v_4^2, \tag{8}$$

$$2\frac{R_4 S_4}{RS} + \left(\frac{S_4}{S}\right)^2 + \frac{1}{S^2} = \rho + \frac{1}{2}v_4^2. \tag{9}$$

The suffix 4 after R, S and v denotes ordinary differentiation with respect to t.

Equations (7)–(9) are three equations in five unknowns R, S, ρ, p and v. For complete determinacy of the system, two extra conditions are needed. One way is to use an equation of state. The other alternative is to assume a mathematical relation on the space time and then to discuss the physical nature of the universe. In this paper, we confine ourselves to the case $p=\rho$ which describes several important cases, e.g. radiation, relativistic degenerated Fermi gas [18], and probably, superdense degenerated baryon matter at low temperature [16, 17, 19]. The equation of state for ideal gas has also this form in causal limit. Therefore, we take

$$p=\rho \tag{10}$$

and

$$R=AS^n, \tag{11}$$

where 'A' and 'n' are both constants.

From (7)–(10), we get

$$\frac{S_{44}}{S}+\frac{R_4S_4}{RS}+\left(\frac{S_4}{S}\right)^2+\frac{1}{S^2}=0, \tag{12}$$

$$\frac{R_{44}}{R}+\frac{S_{44}}{S}+3\frac{R_4S_4}{RS}+\left(\frac{S_4}{S}\right)^2+\frac{1}{S^2}=0. \tag{13}$$

From (12) and (13), we get

$$\frac{R_{44}}{R}+2\frac{R_4S_4}{RS}=0. \tag{14}$$

Using (11), (14) becomes

$$n\frac{S_{44}}{S}+n(n+1)\left(\frac{S_4}{S}\right)^2=0. \tag{15}$$

Solving (15), we obtain

$$S=N(k_1t+k_2)^{\frac{1}{n+1}}, \tag{16}$$

where $N=(n+2)^{\frac{1}{n+2}}$.

Using (16) in (11), we get

$$R=M(k_1t+k_2)^{\frac{n}{n+2}}, \tag{17}$$

where $M=AN^n$.

From (6), we obtain

$$v_{44}+v_4\left[\frac{R_4}{R}+2\frac{S_4}{S}\right]=0. \tag{18}$$

From (16)–(18), we get

$$\begin{aligned}&\frac{v_{44}}{v_4}+\frac{k_1}{k_1t+k_2}=0,\\&\frac{v_{44}}{v_4}+\frac{k_1}{T}=0\quad(\text{where } k_1t+k_2=T).\end{aligned} \tag{19}$$

Integrating, we get

$$v_4 T = k_3, \tag{20}$$

$$\therefore\ v = k_4 \log T + k_5 \quad \left(\text{where } k_4 = \frac{k_3}{k_1}\right) \tag{21}$$

here k_3, k_4, k_5 are constants of integration.

Using (16) and (17), the line element (1) becomes

$$ds^2 = dt^2 - M^2(k_1 t + k_2)^{\frac{2n}{n+2}} dr^2 - N^2(k_1 t + k_2)^{\frac{2}{n+2}} (d\theta^2 + \mathrm{Sin}^2\,\theta d\phi^2). \tag{22}$$

This metric can be transformed through a proper choice of coordinates to the form

$$ds^2 = dT^2 - M^2(T)^{\frac{2n}{n+2}} dr^2 - N^2(T)^{\frac{2}{n+2}} (d\theta^2 + \mathrm{Sin}^2\,\theta d\phi^2). \tag{23}$$

3 Some Physical and Geometrical Properties

The pressure 'p' and density 'ρ' in the model (23) are given by

$$p = \rho = \frac{(2n+1)}{(n+2)^2}\frac{k_1^2}{T^2} + \frac{1}{(n+2)^{\frac{2}{n+2}}(T)^{\frac{2}{n+2}}} - \frac{k_3^2}{2T}, \tag{24}$$

$$\text{Spatial Volume} \quad v^3 = \sqrt{-g} = MN^2 T, \tag{25}$$

$$\text{Scalar expansion} \quad \theta = \frac{1}{3}u^i_{;i} = \frac{1}{3T}, \tag{26}$$

$$\text{Shear Scalar} \quad \sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = 1/18T. \tag{27}$$

Deceleration parameter [23]

$$q = \frac{-3}{\theta^2}\left[\theta_\alpha u^\alpha + \frac{1}{3\theta^2}\right] = 10 > 0. \tag{28}$$

The model (23) has no initial singularity, while the energy density ρ, pressure p given by (24) possess initial singularities. However, as T increases these singularities vanish. The spatial volume of the model given by (25) shows the anisotropic expansion of the universe (23) with time. For the model (23), the expansion scalar θ and shear scalar σ tend to zero as $T \to \infty$. The positive value of the deceleration parameter indicates that the model decelerates in the standard way.

Also, since

$$\lim_{T\to\infty}\left(\frac{\sigma}{\theta}\right) \neq 0. \tag{29}$$

The model does not approach isotropy for large values of T.

4 Conclusion

We have studied cosmological model generated by perfect fluid coupled with massless scalar field for Kantowski-Sachs space time in general theory of relativity. The model is free from singularities and it is expanding, anisotropic and decelerates in the standard way. Also we find that all the physical quantities like pressure and density diverges at the initial moment of creation.

References

1. Mohanty, G., Tiwari, R.N., Rao, J.R.: Int. J. Theor. Phys. **21**, 105 (1982)
2. Weber, E.: J. Math. Phys. **25**, 3279 (1984)
3. Weber, E.: J. Math. Phys. **26**, 1308 (1985)
4. Lorenz, D.: J. Phys. A: Math. Gen. **16**, 575 (1983)
5. Lorenz, D.: Astrophys. Space Sci. **98**, 101 (1984)
6. Gron, P.: Math. Phys. **27**, 1490 (1986)
7. Matravers, D.R.: Gen. Relativ. Gravit. **20**, 279 (1988)
8. Krori, K.D., Goswami, A.K., Purkayastha, A.D.: J. Math. Phys. **36**, 1347 (1995)
9. Dabrowski, H.P.: J. Math. Phys. **36**, 2945 (1995)
10. Li, X.Z., Hao, J.G.: Phys. Rev. D **68**, 083512 (2003)
11. Brans, C.H., Dicke, R.H.: Phys. Rev. **124**, 925 (1961)
12. Singh, T., Agrawal, A.K.: Astrophys. Space Sci. **182**, 289 (1991)
13. Reddy, D.R.K.: Astrophys. Space Sci. **286**, 365 (2003)
14. Reddy, D.R.K.: Astrophys. Space Sci. **286**, 397 (2003)
15. Kantowski, R., Sachs, R.K.: J. Math. Phys. **7**, 443 (1966)
16. Bludman, S.A., Ruderman, M.A.: Phys. Rev. D **1**, 3244 (1970)
17. Zeldovich, Y.B.: Zh. Eksp. Fiz. **41**, 1009 (1961)
18. Zeldovich, Y.B., Novikov, I.D.: Relativistic Astrophysics, vol. 1. University of Chicago Press, Chicago (1971)
19. Walecka, J.D.: Ann. Phys. **83**, 491 (1974)
20. Panigrahi, U.K., Sahu, R.C.: Theor. Appl. Mech. **30**(3), 163–175 (2003)
21. Saez, Ballester, V.J.: Phys. Lett. A **113**, 467 (1985)
22. Gasperini, M.: Phys. Rep. **373**, 1 (2003)
23. Finstein, A., Ibanez, J., Labarge, P.: J. Math. Phys. **36**, 4962 (1995)
24. Roy, Choudhary, A.K.: Theoretical Cosmology. Claredon, Oxford (1979)